The Solow Model

February 5, 2010

"Is there some action a government of India could take that would lead the Indian economy to grow like Indonesia's or Egypt's? If so, what exactly? If not, what is it about "the nature of India" that makes it so? The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else."

- Robert Lucas, Nobel Laureate (1995).

1 Economic Growth

One of the most striking features of the world economy is the vast disparity in standards of living and rates of economic growth. For example, in 2000, real GDP per capita in the United States was more than fifty times that in Ethiopia. And over the period $1975 - 2003$, real GDP per capita in China grew at a rate of 7.6% annually, while, in Argentina, real GDP per capita grew at a rate of only .1%—seventy six times slower. Moreover, there are often vast reversals in prosperity over time. Argentina, Venezuela, Uruguay, Israel, and South Africa were in the top 25 countries (by GDP per capita) in 1960, but none made it to the top 25 in 2000. From 1960 to 2000, the fastest growing country in the world was Taiwan, which grew at 6%. The slowest growing country was Zambia which grew at -1.8%. That is, people in Zambia were markedly worse off in 2000 than they were in 1960. The theory of economic growth seeks to address these issues and provide explanations.

The Solow model is a long-run model that seeks to explain why there are such vast income disparities and growth differences across the world. It describes the evolution of *potential* output or the productive capacities of countries over time. Because we assume an economy is always at potential in the long run, there are no recessions or booms in this analysis. Furthermore, there is no mention of nominal quantities such as money and prices since the *classical dichotomy* holds; namely, money has no effect on output in the long run and is thus irrelevant for explaining output differences. Over the long run, printing pieces of paper cannot generate increases in prosperity.

Before going on, it is important to note that we are not really interested in aggregate income/output, Y, in this world. What we are really concerned with is income *per capita*, $\frac{Y}{L}$—our conventional measure of *standard of living*. This shows how well each of us, on average, is doing.[1] Most importantly, I assume the population of size L is constant; that is, there is no population growth. As a result, when Y changes, $\frac{Y}{L}$ will also rise; and when K rises, $\frac{K}{L}$ rises.

1.0.1 The Aggregate Production Function

The *production function* for the economy is:

$$Y = AK^{\alpha}L^{1-\alpha}, \qquad 0 < \alpha < 1. \tag{1}$$

The production function is our *first key equation.* An important feature of this function is that it exhibits *decreasing returns to scale* to capital. What this means is that a second laptop, say, will not give as much *bang per buck* as the first laptop, and so on. Formally, we say there's *diminishing marginal productivity* to capital accumulation. Keep in mind that we assume all the units of capital (say, laptops) are *the same.* So what I mean by increases in the capital stock are *more of the same* units.

Examining the production function more carefully, we see immediately that differences in GDP are attributable to differences in total factor productivity, A, differences in physical capital, K, and differences in population, L. The A term denotes *total factor productivity*; it is anything (in a broad sense) that, for a given K and L, leads to greater output. It's not simply technological advances, but also encompasses such factors as culture, climate, skills, health, work ethic, social capital (i.e., culture), human capital (education), institutions (the political system etc), and so on. Basically, A measures the efficiency with which K and L are combined. Total factor productivity rises if output per worker increases for any *given* K and L. It could be just people working more hours. For example, in a country with a persistently hot and inhospitable climate, then Y will likely be lower for a given level of capital per worker; hence A is

[1] In the United States, and indeed most industrialized Western countries, this has been growing at an average rate of 1.8% since 1870 or about 2% since 1900.

relatively lower. Just imagine Trinity on a great day—no matter how great your PC is, you'll still get less done; that is, A will fall. Yet, for now, I assume A is a constant at any point in time.

To graph this, just let $\beta = AL^{1-\alpha}$ for a moment, and so the production function becomes

$$Y = \beta K^{\alpha}, \qquad 0 < \alpha < 1. \tag{2}$$

Using this formulation, we can treat β as a constant and just draw a graph of Y against K. Figure 1 shows the result.

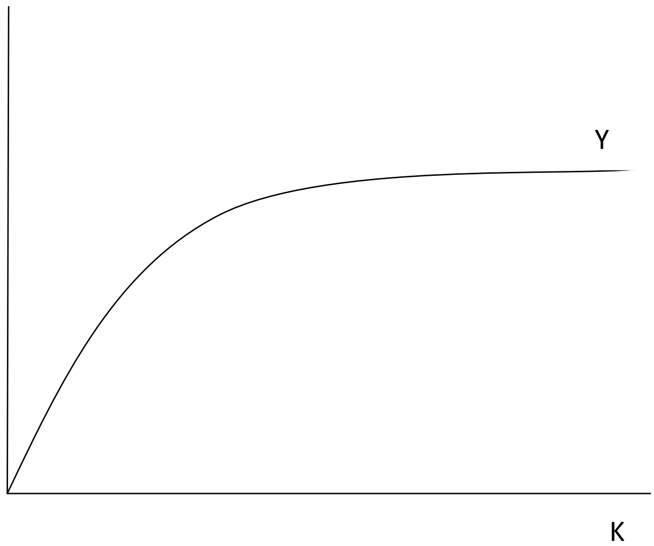

Figure 1: Basic Production Function. The first laptop is extremely useful; the second less so; the third hardly any use at all; and so on. Formally, the marginal product of capital (i.e., its usefulness), falls as K increases. This is called the diminishing marginal product of capital.

The slope of the function subsides as we move outwards. As we shall see, this is what really frustrates things in this economy in the long run. Note crucially that an increase in A will raise Y for any given level of K. Graphically, the way to represent a rise in A is to shift the production function upwards. Figure 2 shows the result. For example, more *powerful* computers would be best represented as an increase in technology, A: just think of each existing computer getting more power via new software.

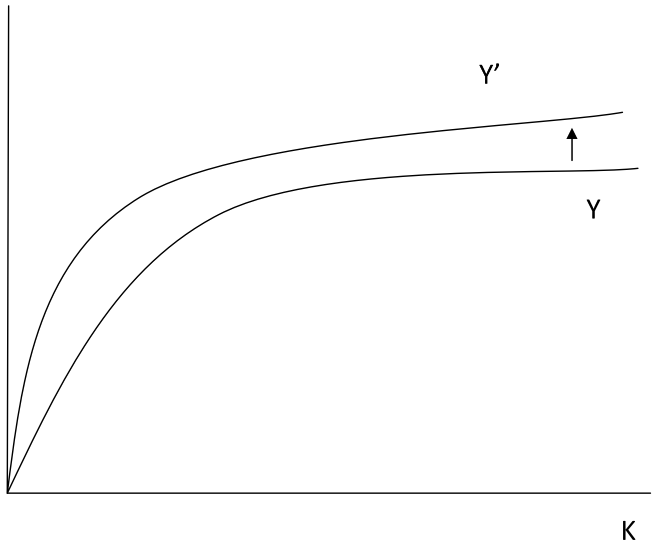

Figure 2: AN INCREASE IN A SHIFTS THE PRODUCTION FUNCTION UPWARDS. SUCH INCREASES IN A ARE A WAY OF OVERCOMING THE CURSE OF DIMINISHING RETURNS.

1.0.2 Growth in A

Because L is fixed, there are only two factors that increase Y, and hence the standard of living, $\frac{Y}{L}$: capital K and total factor productivity A. Because the marginal product of capital tends to zero, increments in capital are little use to us after a certain point. Therefore, in the long run, moving outwards to the right becomes futile, yielding only negligible increases in Y. For this reason, we need something else to generate *sustained* growth of Y and that something else (by a process of elimination) is productivity growth; i.e., growth in A. Without growth in A, we see that growth eventually stalls.[2] If we can't grow by moving out to the right (that is, by capital accumulation), then *the only way is up*; in the long run, the productivity factor, A, *must* grow to shift our function upwards and generate sustained growth. And that's basically what's happening in the industrialized economies. That's the only way we can have sustained rises in income per capita or standard of living. However, notice that capital accumulation is important at the *outset* of development. But once developed (i.e., when diminishing returns are prevalent), capital accumulation is less useful, and now it's productivity that drives sustained growth. Also, capital accumulation is the "easy" part; it's much harder to generate increases in A. Inspiration, not perspiration, is the key to sustained growth. Many countries such as Russia were remarkably successful in

[2] In reality, we *have* sustained growth in Y and $\frac{Y}{L}$; in developed economies such as the United States, for example, $\frac{Y}{L}$ increased tenfold in the U.S. from 1870 to 2000.

accumulating capital (through forced savings, generating so-called "Stalinist growth") and grew rapidly for a while, but couldn't subsequently generate increases in A.

1.0.3 Capital Accumulation

Central to the model is the evolution of the capital stock; once we know the level of capital, we can find every other variable. Thus, our *second key equation* describes the evolution of the capital stock or *capital stock accumulation.*

First, we assume the stock of *savings* in the economy are some constant fraction, $0 < s < 1$ of output Y. This leaves people with $(1-s)Y$ to consume. Savings includes private savings by individuals and public savings by the government; this sum is called *national savings.* For instance, other things constant, government deficits reduce national savings and hence the savings rate. In the model, savings are channeled into investment; i.e., to buy more laptops etc.[3] As a simple example of the saving/investment decision, consider corn: each period you can either eat the corn (i.e., consumption), or place the corn in the ground, and new corn will grow next period; i.e., investment. It is a similar process here. Now, denoting investment by I and aggregate savings by S, we have:

$$S = sY = I$$

where s denotes the national savings rate. That is, savings are used for investment. Figure 3 shows the *savings line.*

Therefore, by raising investment, savings leads to an increase in the capital stock. But unfortunately a fraction $0 < \delta < 1$ of that stock depreciates (or *melts*) each period. Figure 4 shows the *depreciation line.* Overall, the equation for the evolution of the aggregate capital stock, K, is:

$$\Delta K = I - D = sY - \delta K \tag{3}$$

where D denotes depreciation, assumed to be some fraction δ of the capital stock, K. So the net increase in the capital stock is investment minus depreciation. Think of it this way: what determines the increase in *you* is the amount of calories you consume, minus the number of calories you burn up. It is precisely the same idea here.[4]

[3]Note the importance of *financial intermediation* or banks here. It's crucial to have a credible banking system so as to people can save and so as to channel those savings to the right investment projects.

[4]People attempting to lose weight decrease their food intake (i.e., their "investment") and engage in exercise (i.e., trying to create more "depreciation.")

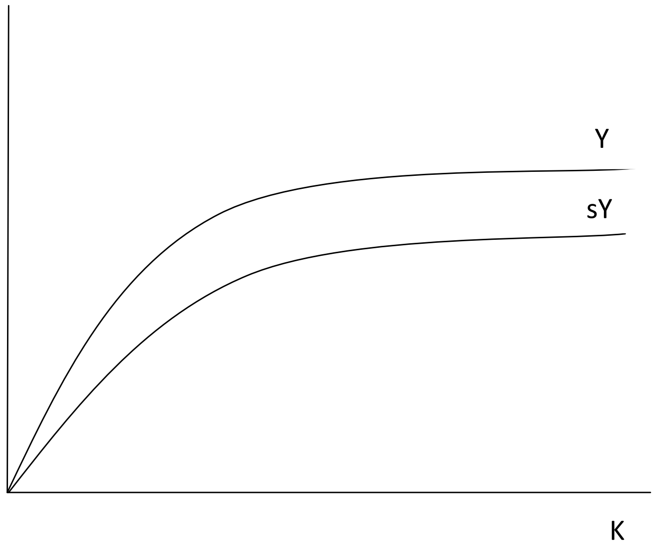

Figure 3: Savings is some fraction s of output. At each level of K, the savings line, sY, indicates what adds to the capital stock.

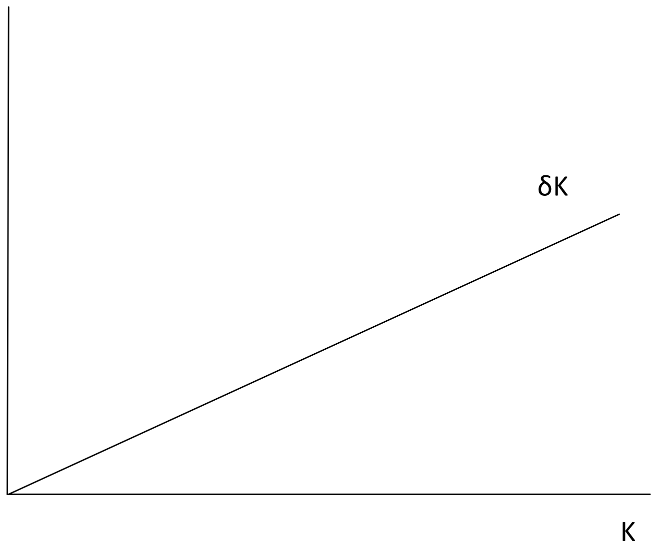

Figure 4: At each level of K, the depreciation line indicates how much capital depreciates in the economy. In other words, this tells us what leaves the capital stock.

At the heart of the model is the tension between investment and depreciation. It is useful to think of these as a tug of war between the "good" and "bad" force. The tension between total depreciation and investment determines the evolution of the capital stock and ultimately the standard of living. But there is bad news. Depreciation is linear in capital. By contrast, the gains to saving are high initially, but fall as diminishing returns to capital set in. As a result, we must reach a point where the depreciation force just offsets the gains to investment and, assuming A is fixed, there is no further growth K or Y. This point is called the *steady state* level of capital, K^*. Associated with this level of capital, is the steady state level of output, Y^*.

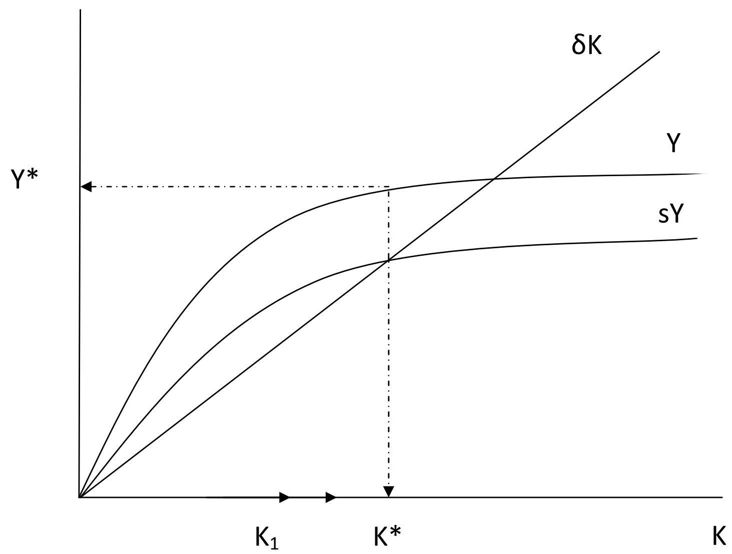

Figure 5: The Complete Solow Model. Once the savings/investment function is above the depreciation line (as at point K_1), the capital stock increases. In contrast, when the depreciation line is above the savings/investment function, the capital stock falls. Point is, the economy will always gravitate to steady state. At the steady state, the two forces just offset each other and the capital stock stalls. It can take an economy at least a decade to go from K_1 to K^*.

1.0.4 The Transitional Path to Steady State

Figure 5 shows the complete Solow model. The steady state capital stock is at the intersection of the savings and depreciation lines. When the economy is the the left of the steady state, investment exceeds depreciation. As a result, the capital stock K must increase. And given that K is increasing as it moves towards steady state, we know from the production function, $Y = AK^{\alpha}L^{1-\alpha}$, that Y is necessarily increasing too. When the savings/investment function and the depreciation line intersect, then total depreciation and investment just offset each other. Thereafter, absent growth in A, there is no further growth in K or Y. The economy is now at *steady state* and in its *golden years*. At this point, growth in A must drive sustained rises in the standard of living.[5]

To summarize, we have two key equations that describe the evolution of the economy. They are:

$$\boxed{Y = AK^{\alpha}L^{1-\alpha}} \tag{4}$$

and

$$\boxed{\Delta K = sY - \delta K} \tag{5}$$

So when the savings line is above the depreciation line, mathematically we have, $sY > \delta K$, and hence from Equation (5), $\Delta K > 0$. Then from Equation (4), output will also increase. At the steady state level of capital $\Delta K = 0$, and hence from Equation (4), output, Y, will also stop growing.

Key Idea 1 *If the savings/investment function is above the depreciation line, then the capital stock and output are rising. The opposite applies if the savings/investment function is below the depreciation line.*

1.0.5 The Steady State

It's important to bear in mind that steady state Y and hence $\frac{Y}{L}$ (i.e., welfare/standard of living) is constant and determined by the exogenous parameters of the model: s, δ, A. While increases in s and A lead to higher standard of living, increases in δ lowers it. Increases in s and A ultimately result in more savings and investment, and so lead to a higher steady state capital stock.[6]On the other hand, a higher rate of depreciation causes a greater losses in capital stock, which ultimately lowers the steady state level of capital. Because countries across the world have different values of s, δ, and A, they converge to different steady states. Also, just because an economy is in equilibrium, it does not mean the equilibrium is good; rather, it could settle at a point of impoverishment. Once a country reaches its steady state, we have no further growth in K or Y, without growth in A. So, without A growing, we have no *sustained* growth in output per worker in steady state. All of the growth we see in the model is thus *transitional* and occurs on the *transition* to steady state. To ensure continually rising living standards, A must rise. Such continuous rises in A are attributable to technological progress. While a better political system would lead to a once-off increase in A, it is unlikely that it would lead to a continuous rise in A over time.

[5]You could continue to increase the savings rate, but the resulting increases in output would be tiny.

[6]Note that a higher A leads to a higher $Y = AK^{\alpha}L^{1-\alpha}$ and so to higher stock of savings, sY.

For this reason, it is technological progress that ultimately drives growth in developed economies. And because technological advances can be "shared" across advanced countries, the model predicts advanced economics should grow at about the same rates—a prediction that is confirmed in the data.

Key Idea 2 *When an economy reaches steady state (or its golden years), there is no further growth in* Y *or* $\frac{Y}{L}$ *(without growth in* A*), since the forces of depreciation just offset the forces of investment. What determines the steady state level of* Y *are the "fundamentals":* s*,* δ*, and* A*. To generate sustained growth in* Y*,* A *must rise continuously.*

1.0.6 Convergence

Each country grows towards its own steady state determined by its fundamentals, A, s, and δ. So clearly, countries with *different* parameters converge to *different* steady states. From the basic Solow diagram in Figure 5, we can see that countries further below their steady states grow faster since $\Delta K > 0$. The economy, in this sense, is analogous to a *spring*—the further the economy is "stretched" from its steady state point, the faster it will grow. Fundamentally, the reason for this is that there countries have not yet encountered severe diminishing returns. For them, the *marginal product of capital* is relatively high, leading to large increases in output.[7] Think of *running* from Pearce St. to the Arts Block; you start off pretty quickly, but this is unsustainable, and your speed slows down as you approach the Arts Building. It's the same intuition for the growth of nations towards their respective steady states.

But wait. What about Africa, is this continent now poised for rapid growth? Not at all. Africa would *not* be expected to grow fast under this rationale, since it has a low steady state due to poor fundamentals (mainly, low A and low s). It's only countries with good fundamentals that are *starting off* poor that would be expected to grow faster. What determines growth is the distance between where they are now and their steady state (i.e., where they are going.) This is called *conditional convergence* or *catch-up growth.* Examples of such rapid growth include China and the other Asian "tiger" economies like Taiwan and South Korea. Think of it like this: if your micro result was pretty poor, but you now start working harder (and so improving your "fundamentals") you have the potential from rapid improvement for the macro test. By contrast, those who performed well in micro have little potential for growth in their mark. More significantly, we don't expect any improvement from those who performed poorly in micro, and take no measures to improve.

Key Idea 3 *The further the economy is from its steady state, the faster it will grow.*

[7] The marginal product of capital is the increase in output from adding one more unit of capital.

Key Idea 4 (Conditional Convergence) *Poor countries that are heading to a good steady state can experience rapid growth. Namely, such countries have not yet hit severe diminishing returns, and so output grows quickly has they head towards their steady state.*

1.1 Savings and Development

One central insight from the Solow model is its prediction for what happens when the savings rate changes. So suppose the savings rate, s, changes due to a rise in personal savings. The investment/saving function shifts up, and we initially grow towards a new high steady state. As we move towards the new steady state, K and hence Y rise. In the new steady state K and Y are both higher and remain constant at a higher rate. Significantly, there is no effect on the long-run growth of K and Y. It was only along the *transition path* to steady state that we had a spurt of growth. But that growth effect is ultimately transitory and leads us merely to a higher *level* of Y in the long run. What is striking is we don't get a permanently higher growth rate of Y. Underlying this important result is the principle of *diminishing returns*. More savings just means more investment, but this is not of *that* much use to us. Because all the antecedents to the Solow model predicted that the long-run growth of Y was *increasing* in the savings rate, Solow himself described this as a "real shocker" Finally, keep in mind that the time to complete the transitional path to a new steady state is long in this model—probably decades—so we can actually have *transitional growth effects* for quite a while. Finally, empirical work shows that the elasticity of output with respect to the saving rate is about .5. That is, a 10% increase in the saving rate raises the level of output per person in the long run by about 5%, only a modest increase.

To understand these effects, consider body weight. One often reads something to the effect that eating an extra bar of chocolate a day will cause you to put on, say, 2 pounds a week. Just think about that for a moment. After a year you'll put on 104 pounds, and after two years you'll put on 208 pounds, and so on. Fortunately, this is self-evidently *wrong*; when you increase your consumption rate of chocolate, there cannot realistically be a *long-run growth effect*. Most likely, you will grow for a while in size—i.e., a *transitory growth effect*, to be sure— <u>but</u> you will asymptote eventually to some greater weight/*level*. This prediction seems more reasonable. In other words, there's a transitory growth effect and long-run effect on your weight level. In particular, you wont keep growing *forever*. Of course, this is analogous to what happens when the savings rate increases in the Solow model.

1.2 Examples

Getting back to macro, here are three examples of changes to the model. In all cases, there are three mains steps to figuring out how Y (and hence $\frac{Y}{L}$) changes. First, you figure out which curve shifts. Second, figure out where the new intersection of the savings and depreciation line is. This will nail down the new steady state capital stock. Third, from the new steady state capital stock, go up to the production function to find the new steady state level of Y.

1.2.1 Example 1: An Fall in the Savings Rate

Suppose we are at the steady state K^* and there is a sudden and permanent fall in the savings rate from s to s' time t. This is shown in Figure 6. This change causes the savings line to shift downwards. We start off at K^*. Just after the change, then, we now have the depreciation line above the new savings line. As a result, the capital stock falls, and will continue to fall until the depreciation line intersections with the new savings line. This point of intersection is the new steady state K', where the associated output level is Y'.

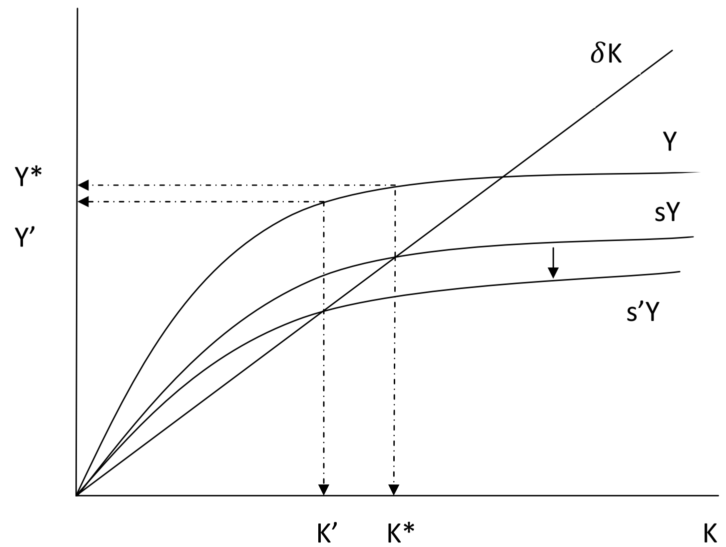

Figure 6: A fall in the savings rate from s to s' in the Solow Model.

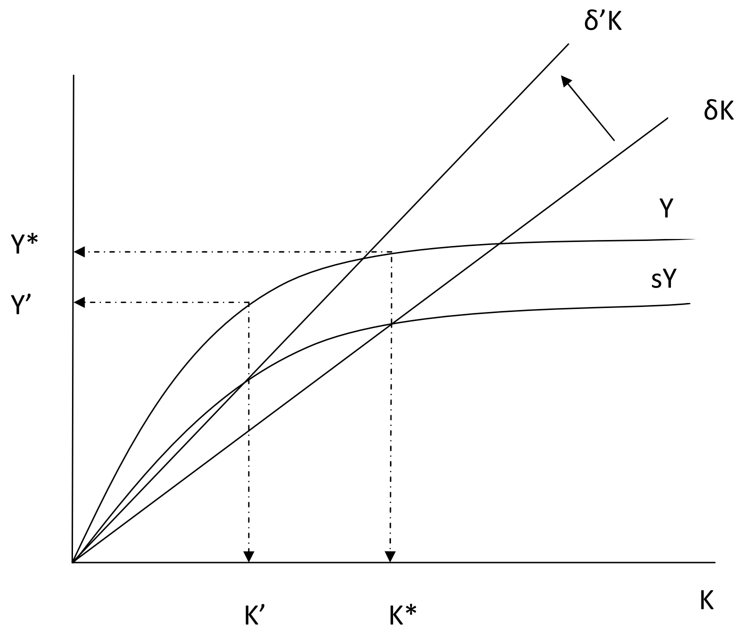

Figure 7: An Increase in Depreciation from δ to δ' in the Solow Model.

1.2.2 Example 2: A Rise in the Depreciation Rate

Figure 7 illustrates a rise in the depreciation rate. A rise in the rate of depreciation from δ to δ' causes an increase in the slope of the depreciation line. As a result, the line pivots upwards. The intersection of this line with the the savings line determines the new steady state level of capital, K', which is lower than before. The associated steady state output Y' is also lower.

1.2.3 Example 3: A Rise in Total Factor Productivity, A

A rise in A has two effects. First, it causes the production function to shift upwards: A rise in A causes $Y = AK^{\alpha}L^{1-\alpha}$ to rise for any given level of K i.e., the production function shifts up. Second, because savings is a constant fraction of Y, the savings stock is now a constant fraction of a *higher* Y. That is, savings moves from being sY to sY'. As a result, the savings line shifts upwards. The intersection of this new savings line with the depreciation line pins down the new steady state capital stock, K'. As noted already, it is such increases in A that ultimately cause sustained increases in living standards in developed economies.

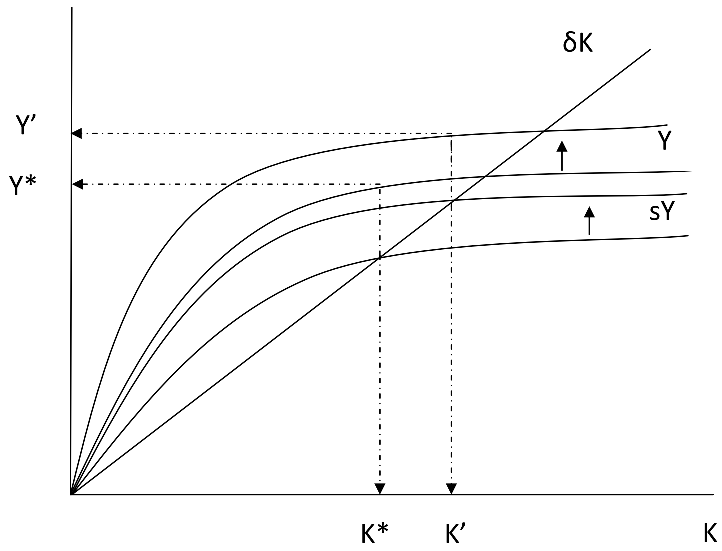

Figure 8: An increase in A. Ultimately, it is continual increases in A that sustain increases in living standards in developed countries. Observe how an increase in A induces an increase in K too.